Combinatorial optimization for undergraduate students

Lecture note 1. Basic graph theory

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A graph is a *pair* $(V(G), E(G))$ of a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$. We shortly denote uv for an edge (u, v). We say that a vertex u is adjacent to v if $uv \in E(G)$. We also say that u is a neighbor of v.

A graph H is an induced subgraph of a graph G if H can be obtained from G by deleting vertices (with incident edges). A graph H is a subgraph of a graph G if H can be obtained from G by deleting vertices and edges.

For a vertex v and a vertex subset S, we denote by $G - v$ the graph obtained by removing v, and denote by $G - S$ the graph obtained by removing S. For an edge e and an edge subset F, we denote by $G - e$ the graph obtained by removing e, and denote by $G - F$ the graph obtained by removing F.

A sequence $v_0 - e_1 - v_1 - e_2 - v_2 \cdots - e_n - v_n$ is called a walk if for every i, e_i is incident with v_{i-1} and v_i. If all of e_i's are distinct, then it is called a trail. It is a closed trail if $v_0 = v_n$. If no two vertices are repeated, then it is called a path. A closed trail for which all v_i's are all distinct except $v_0 = v_n$ is a cycle.

Exercise 1. *Every walk from a to b contains a path from a to b.*

Two vertices a and b of a graph G are called *connected* if there exists a path from a to b. A graph is connected if all pairs of vertices are connected. A graph that is not connected is called disconnected. Each maximal subgraph that is connected is called a connected component of the graph.

Lemma 1. *Every connected graph on n vertices has at least $n-1$ edges.*

Königsberg bridge problem: Given a graph, is there a trail that uses all edges?

A multigraph is a graph that allow multi-edges $(u,v)_1,(u,v)_2,\ldots,(u,v)_m$ between two vertices u,v. For a vertex v, the *degree* of a vertex v is the number of edges incident with v.

An *Eulerian trail* of a multigraph G is a trail which contains all edges of G, and an *Eulerian tour* is an eulerian trail which is a closed trail. A graph is *Eulerian* if it has an Eulerian tour.

Theorem 1. *Let G be a connected multigraph. The following are equivalent.*

(1) G is Eulerian
(2) Each vertex of G has even degree.

Travelling salesman problem : Consider the graph whose edges are weighted by some integers (it may correspond to distance between two points for instance). Can we find efficiently a cycle of a graph that covers $V(G)$ and uses minimum weight?

This problem turns out to be difficult (which we will say NP-hard).

Looking at all possibilities for tours in a complete graph would be a lot of work: for instance, if there are 9 vertices, then we need to consider $8!/2 = 20160$ possiblities. Note that $n! = 2^{\Omega(n \log n)}$.

A *hamiltonain cycle* is a cycle that go though all vertices.

A graph is a planar graph if it can be drawn in the plain without crossing edges.

Very famous non-planar graphs are K_5 and $K_{3,3}$.

A graph H is a *minor* of another graph G if H can be obtained from G by a sequence of deleting vertices, deleting edges, and contracting edges. It is easy to see (by picture) that if H is a minor of G and G is planar, then H is also planar.

Theorem 2 (Wagner's theorem, without proof)**.** *A graph is planar if and only if it contains no K_5-minor and $K_{3,3}$-minor.*

Theorem 3 (Euler's formula)**.** *Let G be a connected planar graph with n vertices, m edges and f faces. Then $n - m + f = 2$.*