Combinatorial optimization for undergraduate students

Lecture note 10. Flow-The algorithm of Edmonds and Karp

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The labelling algorithm of Ford and Fulkerson is not polynomial in general. Edmonds and Karp modify the algorithm so that it runs in time $\mathcal{O}(|V(G)||E(G)|^2)$ [Theoretical improvements in algorithmic efficiency for network flow problems. J. Assoc. Comput. Mach 1972].

We will see that it suffices if we always use an augmenting path of shortest length.

Theorem 1. *Replace the step*

(5) *choose a vertex v which is labelled and satisfies $u(v) = false$;*

as follows:

(5') *among all vertices with $u(v) = false$, let v be the vertex which was labelled first.*

Then the resulting algorithm has complexity $\mathcal{O}(|V(G)||E(G)|^2)$.

Proof continue

Example.

Auxiliary networks and phases.

Let $N = (G, c, s, t)$ be a flow network with a flow f. We define a new flow network (G', c', s', t') as follows:

- $V(G') = V(G)$,
- for each edge $e = uv$ of G with $f(e) < c(e)$, there is an edge $e' = uv$ of G' with $c'(e') = c(e) - f(e)$,
- for each edge $e = uv$ with $f(e) \neq 0$, G' contains an edge $e'' = vu$ with $c'(e'') = f(e)$.

It is called the auxiliary network with respect to f.

In the algorithm of Ford and Fulkerson, steps (6) to (9) for forward edges and in steps (10) to (13) for backward edges, use only those edges e of G for which G' contains e' or e''. And a augmenting path W with respect to f in G corresponds to a uniquely determined directed path W' from s to t in G'.

We can use G' to decide whether f is maximal and if not, to find an augmenting path.

Lemma 1. *f is maximal if and only if t is not accessible from s in G'.*

It is a translation of Augmenting path theorem

Lemma 2. *Let N be a flow network with a flow f, and let N' be the corresponding auxiliary network. Moreoever, let f' be a flow on N'. Then there exists a flow f'' of value $w(f'') = w(f) + w(f')$ on N.*

Theorem 2. *Denote the value of a maximum flow on N and on N' by w and w', respectively. Then $w = w' + w(f)$.*