Combinatorial optimization for undergraduate students

Lecture note 7. Spanning trees-Steiner tree and variations

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Assume that we want to connect n points in the Euclidean plane by a network of minimal total length. We may just view the given points as the vertices of a complete graph. However, it should be possible to do better if we are willing to add some new vertices.

Jacob Steiner studied this problem, and Fermat studied when $n = 3$.

The Euclidean Steiner tree problem is NP-hard.

In this section, we consider a graph-theoretic version, the **Steiner network problem**. One is given a network (G, w) with a positive weight function w, where $R \subseteq V(G)$. The problem asks to find a minimal tree T which contains all vertices in R.

Interestingly, it generalizes both Shortest Path problem and Spanning Tree problems;

- If $R = V(G)$, then it is the Spanning Tree problem.
- If $|R| = 2$, then it is the Shortest Path problem.

The general Steiner network problem is NP-hard.

We will give an algorithm that is exponential in $r = |R|$ but polynomial in $s = |V(G) \setminus R|$. Let $S := V(G) \setminus R$.

Before presenting algorithm, we look at the metric case where G is complete and w satisfies the triangle inequality. Then we will discuss how to reduce the general case to the metric case.

For a steiner tree T, vertices in $V(T) \setminus R$ are called Steiner points.

Lemma 1. *Let G be a complete graph whose vertex set is the disjoint union $V(G) = R \cup S$ of two disjoint subsets. Let w be a positive weight function on $E(G)$ satisfying the triangle inequality. Then there is a minimal Steiner tree that contains at most $|R| - 2$ Steiner points.*

Lemma 2. *Let G be a graph whose vertex set is the disjoint union $V(G) = R \cup S$ of two disjoint subsets. Let w be a positive weight function and d the distance function. Then the weight of a minimal Steiner tree for (K_V, d) is the same as the weight of a minimal Steiner tree for (G, w).*

Let G be a connected graph with $V(G) = \{1, 2, \ldots, n\}$ and a positive weight function $w : E(G) \to \mathbb{R}$ and $V(G) = R \cup S$ for two disjoint subsets. Write $r = |R|$.

We assume that $\mathrm{DIST}(G, w; d, p)$ outputs distance function $d(a, b)$ and shortest paths $p(a, b)$ between all two vertices. (We can use Dijkstra's algorithm for it.)

Algorithm 1 Steiner tree

procedure STEINER$(G, R, w; T)$
    $W \leftarrow \infty, T \leftarrow \emptyset, H \leftarrow K_n$;
    $\mathrm{DIST}(G, w; d, p)$;
    for $i = 1$ to $r - 2$ do
        for $S' \subseteq S$ with $|S'| = i$ do
            $\mathrm{PRIM}(H|(R \cup S'), d; T', z)$;
            if $x < W$ then $W \leftarrow z, T \leftarrow T'$;
    for $e = uv \in T$ do
        if $e \notin E(G)$ or $w(e) > d(u, v)$
        then replace e in T by the edges of a shortest path from u to v in G
end procedure

Theorem 1. *Algorithm 1 constructs a minimal Steiner tree for $(G, R; w)$ with complexity $\mathcal{O}(|V(G)|^3 + 2^{|S|}|R|^2)$.*

Theorem 1 shows that Algorithm 1 is polynomial in $|V(G)|$ for fixed $s = |S|$.

Spanning trees with restrictions: In reality, most of the problems, one encounters cannot be solved by determining just any spanning tree; usually, the solution will have to satisfy some further restrictions. This often leads to much harder problems (often even to NP-hard).

The following restrictions are NP-complete.

- (Degree constrained spanning tree) Let G be a connected graph and k be a positive integer. Is there a spanning tree with maximal degree at most k?

- (Maximum leaf spanning tree) Let G be a connected graph and k be a positive integer. Is there a spanning tree having at least k leaves?

- (Minimum leaf spanning tree) Let G be a connected graph and k be a positive integer. Is there a spanning tree having at most k leaves?

- (Isomorphic spanning tree) Let G be a connected graph and T a tree. Does G have a spanning tree isomorphic to T?

Maximum leaf spanning tree problem remains NP-complete even if we restrict our attention to cubic graphs, that are graphs whose all vertices have degree exactly 3.

Bonsma and Zickfeld [A 3/2-approximation algorithm for finding spanning trees with many leaves in cubic graphs. SIAM J. Discrete Math 2011] contains an efficient approximation algorithm for the problem.

Some weighted version with restrictions. Both problems are NP-complete.

- (Bounded diameter spanning tree) Let G be a connected graph with a weight function $w : E(G) \to \mathbb{N}$ and d, k be positive integers. Is there a spanning tree T with $w(T) \le k$ and diameter at most d?

- (Optimum communication spanning tree) Let G be a connected graph with a weight function $w : E(G) \to \mathbb{N}$ and a request function $r : E(G) \to \mathbb{N}$ and k be a positive integer. Does G have a spanning tree T satisfying
$$\sum_{(u,v)\in V(G)\times V(G)} d(u,v) \times r(u,v) \le k?$$
In practice, $d(u,v)$ signifies the cost of the path from u to v, and $r(u,v)$ is the capacity we require for communication. Then the product $d(u,v)r(u,v)$ is the const of communication between u and v.

The special case of Optimum communication spanning tree where all weights are equal, can be solved in time $\mathcal{O}(|V(G)|^4)$ due to Hu [Optimum communication spanning trees. SIAM J. Computing 1974].