Combinatorial optimization for undergraduate students

# Lecture note 8. Matroid - The greedy algorithm and matroids

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We study a generalization of the algorithm of Kruskal, so called greedy algorithm.

The greedy strategy is rather short-sighted: we always select the element which seems best at the moment. In general, this simple strategy will not work, but for a certain class of structures playing an important part in combinatorial optimization, so called matroids.

We recall the algorithm of Kruskal.

We consider $\mathcal{S}$ as the set of all subsets of $E$ that form a forest. We can think of Kruskal's algorithm as that an edge which is currently examined is added to current $T$ if and only if $T \cup \{e\}$ is also in $\mathcal{S}$. This scheme will be generalized.

Definition of an independence system.

Let $(E, \mathcal{S})$ be an independence systme and $w : E \to \mathbb{R}^+$ be a weight function.

**Algorithm 1** Greedy algorithm

**procedure GREEDY**$(E, \mathcal{S}, w; T)$
    order the elements of $E$ according to their weight: $E = \{e_1, e_2, \ldots, e_m\}$ with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$;
    $T \leftarrow \emptyset$;
    for $k = 1$ to $m$ do
        if $T \cup \{e_k\} \in \mathcal{S}$, then $T \leftarrow T \cup \{e_k\}$
**end procedure**

In the last section, we saw that this greedy algorithm solves the problem of finding a maximum-weighted spanning tree.

Unfortunately, this simple strategy does not work.

An independent system $(E, \mathcal{S})$ is a *matroid* if the greedy algorithm solves the associated optimization problem correctly.

**Corollary 1.** *Let $G$ be a graph and let $\mathcal{S}$ be the set of subsets of $E(G)$ which are forests. Then $(E, \mathcal{S})$ is a matroid.*

The matroid described above is called the *graphic matroid* of the graph $G$.

**Theorem 1.** *Let $M = (E, \mathcal{S})$ be an independent system. Then the following conditions are equivalent:*

*(1)* *$M$ is a matroid.*
*(2)* *For $J, K \in \mathcal{S}$ with $|J| = |K| + 1$, there exists some $a \in J \setminus K$ such that $K \cup \{a\}$ is also in $\mathcal{S}$.*
*(3)* *For every subset $A$ of $E$, all maximal independent subsets of $A$ have the same cardinality.*

Proof continue

**Theorem 2.** *Let $E$ be a finite subset of a vector space $V$, and let $\mathcal{S}$ be the set of all linearly independent subsets of $E$. Then $(E, \mathcal{S})$ is a matroid.*

This matroid is called a vector matroid.

An abstract matroid is called *representable over* $\mathbb{F}$ for some field $\mathbb{F}$ if it is isomorphic to a vector matroid $V$ over $\mathbb{F}$.

**Exercise 1.** *Every graphic matroid is representable over the binary field $\mathbb{F}_2$.*

We show that the greedy algorithm actually works for arbitrary weight functions on a matroid (in the definition, weight was non-negative).

**Theorem 3.** *Let $M = (E, \mathcal{S})$ be a matroid, and let $w : E \to \mathbb{R}$ be any weight function on $M$. Then the greedy algorithm finds an optimal solution for the problem (BMAX) of determining*

$$\max\{w(B) : B \textit{ is a basis of } M\}.$$

**Theorem 4.** *Let $M = (E, \mathcal{S})$ be a matroid, and let $w : E \to \mathbb{R}$ be any weight function on $M$. Then the greedy algorithm finds an optimal solution for the problem (BMIN) of determining*

$$\min\{w(B) : B \textit{ is a basis of } M\}$$