Combinatorial optimization for undergraduate students

Lecture note 9. Flow-The Theorems of Ford and Fulkerson

Lecturer : O-joung Kwon

Spring, 2018

We study flows in networks: How much can be transported in a network from a sourse s to a sink t if the capacities of the connections are given? Such a network might model a system of pipelines, a water supply system, or a system of roads.

Let G be a digraph, and let $c : E(G) \to \mathbb{R}^+$ be a mapping. The value $c(e)$ will be called the capacity of the edge. Let s and t be two vertices, called source and sink, where t is reachable from s.

An *admissable* flow, or shortly a *flow*, on N is a mapping $f : E(G) \to \mathbb{R}^+$ satisfying the following:

(F1) $0 \le f(e) \le c(e)$ for each edge e,

(F2) $\sum_{e^+=v} f(e) = \sum_{e^-=v} f(e)$ for each vertex $v \neq s, t$, where e^- and e^+ denote the start and end vertex of e, respectively.

(F1) is called the feasible condition, and (F2) is called the flow conservation condition.

Lemma 1. *Let $N = (G, c, s, t)$ be a flow network with flow f. Then*

$$\sum_{e^-=s} f(e) - \sum_{e^+=s} f(e) = \sum_{e^+=t} f(e) - \sum_{e^-=t} f(e).$$

The value in Lemma 1 is called the *value* of f, denoted by $w(f)$.

Let $N = (G, c, s, t)$ be a flow network. A flow f is *maximum* if $w(f) \geq w(f')$ for every flow f' on N.

We establish an upper bound for the value of a flow.

A *cut* of N is a partition (S, T) of $V(G)$ into two disjoint sets S and T with $s \in S$ and $t \in T$. The *capacity* of a cut (S, T) is defined as

$$c(S,T) = \sum_{e^- \in S, e^+ \in T} c(e).$$

The cut (S, T) is *minimum* if $c(S, T) \leq c(S', T')$ for every cut (S', T'). An edge e is called *saturated* if $f(e) = c(e)$, and it is *void* if $f(e) = 0$.

Lemma 2. *Let $N = (G, c, s, t)$ be a flow network, (S, T) be a cut, and f be a flow. Then*

$$w(f) = \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e).$$

In particular, $w(f) \leq c(S, T)$, and equality holds if and only if

- *each edge e with $e^- \in S$ and $e^+ \in T$ is saturated, whereas each edge e with $e^- \in T$ and $e^+ \in S$ is void.*

The main result of this section is that the maximum value of a flow equals the minimum capacity of a cut.

A path W from s to t is an *augmenting path* with respect to f if $f(e) < c(e)$ holds for every forward edge $e \in W$, and $f(e) > 0$ holds for every backward edge $e \in W$.

We prove three fundamental theorems due to Fold and Fulkerson [Maximal flow through a network. Can. J. Math 1956].

Theorem 1 (Augmenting path theorem)**.** *A flow f on a flow network is maximum if and only if there are no augmenting paths with respect to f.*

Corollary 1. *Let f be a flow and let S_f be the set of all vertices accessible from s on an augmenting path with respect to f, and put $T_f = V(G) \setminus S_f$. Then f is a maximum flow if and only if $t \in T_f$ (in this case (S_f, T_f) is a minimum cut).*

Theorem 2 (Integral flow theorem)**.** *Suppose all capacities* $c(e)$ *are integers. Then there is a maximum flow on* N *such that all values* $f(e)$ *are integral.*

Theorem 3 (Max-flow min-cut theorem)**.** *The maximum value of a flow on* N *is equal to the minimum capacity of a cut for* N*.*

The remainder of this whole chapter deals with several algorithms for finding a maximum flow.

The proof of Integral flow theorem suggests the following outline of such algorithm:

---

procedure OUTLINE$(N; f, S, T)$
    $f(e) \leftarrow$ for all edges e;
    while there exists an augmenting path with respect to f do
        let $W = (e_1, \ldots, e_k)$ be an augmenting path from s to t;
        $d \leftarrow \min(\{c(e_i) - f(e_i) : e_i \text{ is a forward edge in } W\}) \cup (\{f(e_i) : e_i \text{ is a backward edge in } W\})$
        $f(e_i) \leftarrow f(e_i) + d$ for each forward edge e_i;
        $f(e_i) \leftarrow f(e_i) - d$ for each backward edge e_i;
end procedure

We have to specify a technique for finding augmenting paths.

Algorithm 1 Labelling algorithm of Ford and Fulkerson

procedure FORDFULK$(N; f, S, T)$
  for $e \in E(G)$ do $f(e) \leftarrow 0$
  label s with $(-, \infty)$;
  for $v \in V(G)$ do $u(v) \leftarrow false; d(v) \leftarrow \infty$
  repeat
    choose a vertex v which is labelled and satisfies $u(v) = false$;
    for $e \in \{f \in E(G) : f^- = v\}$ do
      if $w = e^+$ is not laballed and $f(e) < c(e)$, then
      $d(w) \leftarrow \min\{c(e) - f(e), d(v)\}$ and label w with $(v, +, d(w))$
    for $e \in \{f \in E(G) : f^+ = v\}$ do
      if $w = e^-$ is not labelled and $f(e) > 0$, then
      $d(w) \leftarrow \min\{f(e), d(v)\}$ and label w with $(v, -, d(w))$
    $u(v) \leftarrow true$;
    if t is labelled
    then let d be the last component of the label of t;
      $w \leftarrow t$;
      while $w \neq s$ do
        find the first component v of the label of w;
        if the second component of the label of w is $+$,
        then set $f(e) \leftarrow f(e) + d$ for $e = vw$
        else set $f(e) \leftarrow f(e) - d$ for $e = wv$
        $w \leftarrow v$
      delete all labels except for the label of s:
      for $v \in V(G)$ do $d(v) \leftarrow \infty; , u(v) \leftarrow false$
  until $u(v) = true$ for all vertices v which are labelled;
  let S be the set of vertices which are labelled and put $T \leftarrow V(G) \setminus S$
end procedure

From the proofs of Theorems, we get the following theorem.

Theorem 4. *Let N be a network whose capacity function c takes only integral (or rational) values. Then Algorithm 1 determines a maximum flow f and a minimum cut (S, T), so that $w(f) = c(S, T)$.*

This algorithm may fail to terminate for irrational capacities. For such a smallest network, see Zwick [The smallest networks on which the Ford-Fulkerson maximum flow procedure may fail to terminate. Theor. Comput. Sci. 1995].

In the next section, we will see that the augmenting paths can be chosen efficiently.

Exercise 1. *Let N be a flow network for which the vertex capacities are likewise restricted: there is a further mapping $d : V(G) \to \mathbb{R}^+$ and f has to satisfy*

(F3) $\sum_{e^+=v} f(e) \leq d(v)$ *for* $v \neq s, t$.

For instance, we might consider an irrigation network where the vertices are pumping stations with limited capacity. Reduce this problem to a problem for an appropriate ordinary flow network.

Exercise 2. *How can the case of several sources and several sinks be treated?*

Exercise 3. *Suppose both (S, T) and (S', T') are minimum cuts for N. Show that $(S \cap S', T \cup T')$ and $(S \cup S', T \cap T')$ are also minimum cuts.*