# Combinatorial optimization for undergraduate students

## Lecture note 11. Flow-Combinatorial applications

Lecturer : O-joung Kwon
Spring, 2018

We prove some central results in combinatorics using the theorems of Ford and Fulkerson about maximum flows. Transversal theory can be developed from the theory of flows on networks; this approach was first suggested by Ford and Fulkerson (62). Compared to usual approach of taking Hall's marriage theorem, this way of proceeding has a advantage: it also yields algorithms for explicit constructions.

We shall study disjoint paths in graphs, matchings in bipartite graphs, transversals, partially ordered sets.

- Disjoint paths : Menger's Theorem.
- Matchings : König's Theorem
- Transverals : The Marriage Theorem
- Dissections : Dilworth's Theorem

**Edge-disjoint paths.** A set of paths in $G$ with start vertex $s$ and end vertex $t$ is called edge-disjoint if not two of these paths share an edge. A subset $A$ of $E$ is called an *edge separator* for $s$ and $t$, if each path from $s$ to $t$ contains some edge from $A$.

**Theorem 1** (Menger's theorem (edge version))**.** *Let $G$ be a graph or digraph, and $s, t$ be two vertices of $G$. Then the maximum number of edge-disjoint paths from $s$ to $t$ is equal to the minimum cardinality of an edge separator for $s$ and $t$.*

Continue

**Vertex-disjoint paths.** A set of paths in $G$ with start vertex $s$ and end vertex $t$ is called *vertex-disjoint* if not two of these paths share a vertex except $s, t$. A subset $X$ of $V(G) \setminus \{s, t\}$ is called a *vertex separator* for $s$ and $t$, if each path from $s$ to $t$ contains some vertex from $X$.

**Theorem 2** (Menger's theorem (vertex version))**.** *Let $G$ be a graph or digraph, and $s, t$ be two non-adjacent vertices of $G$. Then the maximum number of vertex-disjoint paths from $s$ to $t$ is equal to the minimum cardinality of a vertex separator for $s$ and $t$.*

Continue.

**Exercise 1.** *Let $S$ and $T$ be two disjoint subsets. Show that the minimum cardinality of a vertex separator for $S$ and $T$ is equal to the maximum number of vertex-disjoint paths from $S$ to $T$ (no two paths share a vertex).*

**Matching.** A matching in a graph is a set $M$ of edges no two of which have a vertex in common. A vertex cover is a vertex set $S$ such that $G-S$ has no edges. A graph is bipartite with partition $(S,T)$ if the vertex set can be partitioned into $S$ and $T$ such that all edges intersect both $S$ and $T$.

**Theorem 3** (König's theorem)**.** *Let $G$ be a bipartite graph. Then the maximum number of a matching in $G$ equals the minimum cardinality of a vertex cover.*

In a bipartite graph with partition $(S, T)$, a matching $M$ is said to cover $S$ if every vertex of $S$ is contained in some edge of $M$.

**Theorem 4** (Hall's marriage theorem)**.** *Let $G$ be a bipartite graph with partition $(S, T)$. $G$ has a matching covering $S$ if and only if for every subset $U$ of $S$, $|N_G(U)| \geq |U|$.*

**Dissections.** Let $G$ be a graph or digraph. A subset $X$ of the vertex set of $G$ is an *independent set* if no tow vertices in $X$ are adjacent. The maximum cardinality $\alpha(G)$ of an independent set of $G$ is called the *independence number*. $\beta(G)$ is the minimum cardinality of a vertex cover.

**Lemma 1.** $\alpha(G) + \beta(G) = |V(G)|$.

Let $G$ be a digraph. A set of directed paths in $G$ is a dissection if the sets of vertices in these paths form a partition of the vertex set $V(G)$. The minimum number of paths in a dissection is denoted by $\Delta(G)$. A digraph is transitive if it satisfies that

- if there is a directed path from $s$ to $t$, then there is an edge $st$.

**Theorem 5** (Dilworth's theorem)**.** *Let* $G$ *be a transitive acyclic digraph. Then* $\alpha(G) = \Delta(G)$*.*