Combinatorial optimization for undergraduate students

Lecture note 12. Matching-Augmenting paths

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Spring, 2018

This chapter is devoted to the problem of finding maximum matchings in arbitrary graphs. In contrast to the bipartite case, it is not easy to reduce the general case to a flow problem.

We use *augmenting paths* with respect to a given matching M for constructing a matching of larger cardinality. An *alternating path* with respect to a matching M is a path P for which edges contained in M alternate with edges not contained in M. Such a path is an *augmenting path* if its start and end vertex are distinct exposed vertices (vertices that are not matched).

For a matching M and an augmenting path P, we denote by $M \oplus P$ their symmetric differences.

For a vertex v matched by M, we denote by $mate(v)$ the vertex w for which $vw \in M$.

The following is the basis of most algorithms for determining maximum matchings in arbitrary graphs.

Theorem 1 (Augmenting path theorem)**.** *A matching M in a graph G is maximum if and only if there is no augmenting path with respect to M.*

We need an efficient technique for finding augmenting paths.

Lemma 1. *Let G be a graph, M be a matching in G and u be an exposed vertex with respect to M. Moreover, let P be an augmenting path, and $M' = M \oplus P$. If there is no augmenting path with respect to M starting at u, then there is no augmenting path with respect to M' starting at u either.*

The first polynomial algorithm is due to Edmonds (1965), it runs in time $\mathcal{O}(|V(G)|^4)$. Currently a best running time $\mathcal{O}(|V(G)|^{1/2}|E(G)|)$ is achieved by Fremuth-Paeger and Jungnickel (2003) and Goldberg and Karzanov (2004).

We describe an efficient technique for finding augmenting paths. We begin by choosing an exposed vertex r. If there exists an exposed vertex s adjacent to r, then we can extend by adding the edge rs. We may assume it does not happen.

Otherwise, we take r as the start vertex for a BFS and put all vertices $a_1, a_2, \ldots, a_p$ adjacent to r in the first layer. We put all vertices $b_1, b_2, \ldots, b_p$ where $b_i = mate(a_i)$ in the second layer. The next layer consists of vertices that are adjacent to one of b_i. We continue in this manner.

If we encounter an exposed vertex in one of the odd-numbered layers, then we have found an augmenting path. This motivates the following definition: a subtree T of T with root r is called an *alternating tree* if r is an exposed vertex and if every path starting at r is an alternating path. The vertices in layers 0, 2, 4 are called *outer vertices*, and the vertices in layers 1, 3, 5 are called *inner vertices*.

Let x be a vertex in layer $2i$ and let $y \neq mate(x)$ be a vertex (maybe in T) adjacent to x. There are four possible cases.

Case 1. y is exposed and not yet contained in T. Then we have found an augmenting path.

Case 2. y is not exposed, and neither y nor $mate(y)$ are contained in T. Then we put y into layer $2i + 1$ and $mate(y)$ into layer $2i + 2$.

Case 3. y is already contained in T as an inner vertex. Then adding xy to T would create a cycle of even length. As T already contains an alternating path to y, such edges are redundant. (So we can ignore it)

Case 4. y is already contained in T as an outer vertex. Note that adding xy to T creates a cycle of odd length $2k + 1$ for which k edges belong to M.

Cycles in Case 4 are called *blossoms*.

The difficulties arising from Case 4 are avoided in the algorithm of Edmonds by *shrinking blossoms* to a single vertex. At a later point, blossoms which were shrunk may be expanded again.