Combinatorics and graph theory

Lecture note 11. Double-counting - Sperner's lemma.

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Any graph has an even number of odd-degree vertices. Why?

The proof is a typical double-counting argument.

We can sometimes prove the existence of a certain object using double-counting argument.

Let us draw a big triangle in the plane with vertices A_1, A_2, A_3. We divide it arbitrarily into a finite number of smaller triangles such that no triangle may have a vertex insider an edge of any other small triangle.

Let us assign the labels $1, 2, 3$ to the vertices of the big and the small triangles, under the following rule.

- the vertex A_i gets the label i,
- all vertices lying on the edge A_iA_j of the big triangle may be assigned only the label i or j,
- other vertices have any labels.

Proposition 1 (Sperner's lemma)**.** *In the situation described above, a small triangle always exists whose vertices are assigned all the three labels* $1, 2, 3$.

Sperner's lemma is not a toy problem; it is a crucial step in the proof of a famous theorem called *fixed point theorem*.

Let Δ denote a triangle in the plane with vertices $A_1 = (1, 0)$, $A_2 = (0, 1)$, $A_3 = (0, 0)$.

Theorem 1 (2-dimensional Brouwer's fixed point theorem)**.** *Every continuous function* $f : \Delta \to \Delta$ *has a fixed point.*

We introduce a game. It takes place on a board like the one in Figure 7.2 in the book. (Triangulation of a rectangle)

Alice paints nodes gray and Betty black. In the starting point, Alice has the nodes a, c marked, and Betty has the nodes b, d marked.

Alice wins if she managed to mark all nodes of a path from a to c, and Betty's goal is a path from b to d. If a player is supposed to make a move and has no more nodes to mark, then the game ends in a draw.

Proposition 2. *On a board of the given type (the outer face is a square, all inner faces are triangles), a draw is impossible.*