Combinatorics and graph theory

Lecture note 12. Double-counting - Independent systems

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A system M of sets is *independent* if no two different sets A and B satisfy $A \subseteq B$.

Theorem 1 (Sperner's theorem)**.** *Any independent system of subsets of an n-element set contains at most $\binom{n}{\lfloor n/2 \rfloor}$ sets.*

The bound cannot be improved; all subsets of size exactly $\lfloor n/2 \rfloor$ constitute an independent system.

We will see two essentially different proofs.

A chain of subsets of X is a set $A_1, A_2, \dots A_k$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k$.

FIRST PROOF

CONTINUE

Second proof uses the notion of symmetric chains. A chain is symmetric, if it contains one set of size k, one set of size $k+1$, ..., one set of size $n-k$.

Lemma 1. *For any finite set X, the system 2^X has a partition into symmetric chains.*

SECOND PROOF

As a last example, we prove a bound on edges of $K_{2,2}$-free graphs.

Theorem 2. *If a graph G on n vertices contains no subgraph isomorphic to $K_{2,2}$, then it has at most $\frac{1}{2}(n^{3/2} + n)$ edges.*