Combinatorics and graph theory

Lecture note 13. Probability - Finite probability spaces

Lecturer : O-joung Kwon

Fall, 2018

We will see a remarkable mathematical application of probability, namely how mathematical statement can be proved using elementary probability theory.

Definition 1. *By a finite probability space, we understand a pair* (Ω, P)*, where* Ω *is a finite set and* $P : 2^{\Omega} \to [0, 1]$ *is a function assigning a number from the interval* $[0, 1]$ *to every subset of* Ω*, such that*

- $P(\emptyset) = 0$*,*
- $P(\Omega) = 1$*,*
- $P(A \cup B) = P(A) + P(B)$ *for any two disjoint sets* A *and* B *in* Ω*.*

The elements of Ω are called the *elementary events*.

For instance, if the experiment is rolling one fair die, the elementary events would be '1 was rolled', '2 was rolled' and so one. We can denote by $w_1, w_2, \ldots,$

Subsets of Ω are called *events*. For $A \subseteq \Omega$, $P(A)$ is the probability of the event A.

Several important species of finite probability spaces.

Definition 2 (A random sequence of n 0s and 1s). *The elementary events of this probability space are all n-term sequences of 0s and 1s, that is, elements of the set $\{0,1\}^n$, and all elementary events have the same probability. The probability of each event A equals $|A|/2^n$. We denote this by $\mathcal{C}_n$.*

This probability space models a sequence of n coin tosses, for instance. An example of an event is A: "Heads appear exactly 10 times", whose prob is $\binom{n}{10}/2^n$.

Definition 3 (A random permutation). *Elementary events of this probability space are all permutations of the set $\{1,2,\dots,n\}$, and the prob of an event A is equal $|A|/n!$. The set of all permutations is denoted by S_n, and this probability space by $\mathcal{S}_n$.*

PROBLEM : What is the probability that in a random ordering of $\{1,2,\dots,6\}$, 1 precedes 2?

It can be formulated as the event $A = \{\pi \in S_5 : \pi(1) < \pi(2)\}$.

Definition 4 (A random graph)**.** *This probability space, denoted by $\mathcal{G}_n$, has all the possible graphs on vertex set $\{1, 2, \ldots, n\}$ as elementary events, and all of them have the same probability equal to $2^{-\binom{n}{2}}$.*

Examples of events :

- $S =$ *"the graph G is connected "*
- $B =$ *" the graph G is bipartite "*

Comparing such events is often very difficult, but we are usually interested in a very rough estimate. For two mentioned examples, it turns out that for $n \to \infty$, $P(S)$ rapidly tends to 1 while $P(B)$ approaches to 0 very quickly.

This is sometimes expressed as

- a random graph is almost surely connected,
- almost surely, a random graph is not bipartite.

Proposition 1. *A random graph almost surely is not bipartite. That is* $lim_{n \to \infty} P(B) = 0$*.*
PROOF

Independent events. Two events A, B in a probability space are called *independent* if we have

$$P(A \cap B) = P(A)P(B).$$

Independence means that if Ω is divided into two parts, A and its complement, the event B cuts both these parts in the same ratio.

Most often, we encounter independent events in the following situation: The elements of Ω can be viewed as ordered pairs, so Ω models the possible results of a experiment consisting of two experiments, where result of the first experiment cannot influence the result of the second experiment and vice versa.

The space $\mathcal{C}_n$ is a typical source of such situations. Similarly, ' the graph G has at least one triangle on $\{1, 2, \ldots, 10\}$' and ' the graph G contains a cycle on $\{11, 12, \ldots, 20\}$' are independent.

Definition 5. *Events $A_1, A_2, \ldots, A_n \subseteq \Omega$ are independent if for each set of indices $I \subseteq \{1, 2, \ldots, n\}$,*

$$P(\bigcup_{i \in I} A_i) = \Pi_{i \in I} P(A_i).$$