Combinatorics and graph theory

Lecture note 14. Probability - Expectations

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Definition 1. *Let* (Ω, P) *be a finite probability space. By a random variable on* Ω*, we mean any mapping* $f : \Omega \to \mathbb{R}$ *(real numbers).*

Example 1. In $\mathcal{C}_n$, we can define a random variable f_1 such that for a sequence s, $f_1(s)$ is the number of 1's in s.

Example 2. On the probability space $\mathcal{S}_n$ of all permutations of the set $\{1, 2, \ldots, n\}$, we define a random variable f_2 such that $f_2(\pi)$ is the number of left maxima, that is, the number i such that $\pi(i) > \pi(j)$ for all $j < i$.

Definition 2. *Let (Ω, P) be a finite probability space, and let f be a random variable on it. The expectation of f is the real number denoted by $E[f]$ and defined by*

$$E[f] = \sum_{w \in \Omega} P(\{w\}) f(w).$$

If all the elementary events have the same probability, then it is simply $\frac{1}{|\Omega|} \sum_{w \in \Omega} f(w)$.

Example (number of 1's) : The random variable f_1 attains a value k for $\binom{n}{k}$ sequences from $\mathcal{C}_n$. Hence we have

$$E[f_1] = \frac{1}{2^n} \sum_{s \in \{0,1\}^n} f_1(s) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} k.$$

It is known that $\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}$. (We will learn later) Therefore, we deduce that $E[f_1] = \frac{n}{2}$.

The value of $E[f_1]$ can be determined in a simpler way. For each sequecne $s \in \mathcal{C}_n$, we consider the sequence $\bar{s}$ arising from s by exchanging all 0s and 1s and all 1s for 0s. We have $f_1(s) + f_1(\bar{s}) = n$. So,

$$E[f_1] =$$

We introduce a simpler way of computing the expectation.

Definition 3. *Let $A \subseteq \Omega$ be an event in a probability space (Ω, P). By the indicator of the event A, we understand the random variable $I_A : \Omega \to \{0, 1\}$ defined in the following way: $I_A(w) = 1$ for $w \in A$ and $I_A(w) = 0$ otherwise.*

Observation : For any event A, we have $E[I_A] = P(A)$.

By definition we have the following.

Theorem 1 (Linearity of Expectation)**.** *Let f, g be arbitrary random variables on a finite probability space (Ω, P), and let α be a real number. Then we have*

$$E[\alpha f] = \alpha E[f], E[f + g] = E[f] + E[g]$$

EXAMPLES: