Combinatorics and graph theory

Lecture note 16. Order from disorder - Ramsey's theorem

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Six people meet at a party. Some of them know each other, some of them don't. Whatever the party may be, there will always be a triangle or a triple of mutually strangers. This is beginning of the Ramsey's theory.

Let G be a graph.

- $\omega(G)$: the maximum number of vertices that are adjacent to each other
- $\alpha(G)$: the maximum number of vertices that are not adjacent to each other

Theorem 1. *Let G be a graph with at least 6 vertices. Then $\alpha(G) \geq 3$ or $\omega(G) \geq 3$.*

PROOF

Here is a generalization of the theorem 1.

Theorem 2 (Ramsey's theorem for graphs)**.** *Let G be a graph with at least $\binom{k+\ell-2}{k-1}$ vertices. Then $\omega(G) \geq k$ or $\alpha(G) \geq \ell$.*

PROOF

The theorem just proved allows us to introduce the following definition. Let $r(k, \ell)$ denote the minimum natural number n such that every graph with n vertices satisfies $\omega(G) \geq k$ or $\alpha(G) \geq \ell$. The number $r(k, \ell)$ is called a Ramsey number.

Theorem 2 says that

- $r(k, \ell)$ exists for every $k \geq 1$ and $\ell \geq 1$.

Summarize simple values of Ramsey numbers :

- $r(1, \ell) = 1$,
- $r(k, 1) = 1$,
- $r(2, \ell) = \ell$,
- $r(k, 2) = k$,
- $r(3, 3) = 6$.

It is known that $r(4, 4) = 18$. But already the value of $r(5, 5)$ remains unknown, in spite of considerable effort of men and computers.

Lower bound for the Ramsey numbers

We let $r(k) = r(k,k)$. By Theorem 2, we know that

$$r(k) \leq \binom{2k-2}{k-1}.$$

What about lower bounds of Ramsey number?

Theorem 3. *For all $k \geq 3$, we have $r(k) > 2^{k/2}$.*