Combinatorics and graph theory

Lecture note 2. Combinatorial counting-functions

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Question: How many distinct 5-letter words using 26 English alphabet? (ex: ywizp)

In the next proposition, we count mappings from an n-element set to a m-element set.

Proposition 1. *Let N be an n-element set, and let M be an m-element set with $m \geq 1$. Then the number of all possible mappings $f : N \to M$ is m^n.*

Proof:

This is another simple and well-known counting result.

Proposition 2. *For $n \geq 0$, any n-element set X has exactly 2^n subsets.*

Proof:

Proposition 3. *Let $n \geq 1$. Each n-element set has exactly 2^{n-1} subsets of an odd size and exactly 2^{n-1} subsets of an even size.*

The following is a variant for one-to-one mappings.

Proposition 4. *For $n, m \geq 0$, there exist exactly*

$$m(m-1)\cdots(m-n+1)$$

one-to-one mappings of a given n-element set to a given m-element set.

Proof:

A bijective mapping of a finite set X to itself is called a *permutation* of the set X. By the previous result, there are $m!$ permutations of an m-element set X.

For a permutation $p : X \to X$, we draw the elements of the set X as dots, and we draw an arrow from each dot x to the dot $p(x)$. This partitions into groups, and call it cycles.

Using this cycle, we may write a permutation as $((1, 4, 5, 2, 8)(3)(6, 9, 7))$.

Binomial coefficients

Let $n \geq k$ be non-negative integers. The binomial coefficient $\binom{n}{k}$ (read n choose k) is defined by the formula

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2\cdot 1} = \frac{n!}{k!(n-k)!}.$$

The basic combinatorial meaning of the binomial coefficient $\binom{n}{k}$ is the *number of all k-element subsets of an n-element set.*

For convenience, we define that

$$\binom{X}{k}$$

is the set of all k-element subsets of the set X.
For example $\binom{\{a,b,c\}}{2} = \{\{a,b\},\{a,c\},\{b,c\}\}$.

Proposition 5. *For any finite set X, the number of all k-element subsets equals $\binom{|X|}{k}$.*

Proof:

Another basic problem leading to binomial coefficients.

How many ways are there to write a non-negative integer m as a sum of r non-negative integers?

The answer is $\binom{m+r-1}{r-1}$. There can be proved in various ways.

Proof:

Theorem 1 (Binomial theorem). *For any nonnegative integer n, we have*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Proposition 6.

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$