Combinatorics and graph theory

Lecture note 3. Combinatorial countings - Estimates

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Fall, 2018

Sometimes, computing an exact result may be possible but laborious, and sometimes, it is beyond our capabilities. Hence, heading for an estimate instead of the exact result may save us lots of work and considerably enlarge the range of problems we are able to cope with.

Example 1. *n-th harmonic number*

$$H_n = 1 + 1/2 + 1/3 \cdots + 1/n$$

It turns out that there is no way to simplify this sum. We want to get some idea about the behavior of H_n for n growing to ∞.

A simple estimate. We let the k-th group G_k consist of the numbers $1/i$ with

$$\frac{1}{2^k} < \frac{1}{i} \leq \frac{1}{2^{k-1}}.$$

That is

$$G_k = \left\{ \frac{1}{2^{k-1}}, \frac{1}{2^{k-1}+1}, \ldots, \frac{1}{2^k - 1}, \right\}$$

and it contains 2^{k-1} numbers.

Observe that

$$\sum_{x \in G_k} x \leq$$

and

$$\sum_{x \in G_k} x \geq$$

Therefore, we get

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \leq$$

and

$$H_n >$$

We can conclude that H_n does grow to infinity but quite slowly, about as slowly as the ( ) function.

Asymptotic comparision of functions.
In mathematics, functions defined on the natural numbers are usually compared according to their behavior as n tends to infinity. This approach is usually called the *asymptotic analysis* of the considered functions.

Big-Oh notation is important.

Definition 1. *Let f, g be real functions defined on the natural numbers. We denote by $f(n) = \mathcal{O}(g(n))$ if there exist constants n_0 and C such that for all $n \geq n_0$, the inequality $|f(n)| \leq C \cdot g(n)$ holds. We read it as "f is big-Oh of g."*

The notation $f(n) = \mathcal{O}(g(n))$ can be understood as saying that the function f does not grow much faster than g.

For example, we may write $10n^2 + 5n = \mathcal{O}(n^2)$.

The $\mathcal{O}()$ notation often allows us to simplify complicated expression wonderfully. For example, we have

$$(7n^2 + 6n + 2)(n^3 - 3n + 2^8) = \mathcal{O}(n^5).$$

Why?

After some practicing, one can write estimates without too much thinking, by quickly spotting the "main terms" in an expression. Such insight is usually based on a use of the following simple rules.

Fact 1. *Let $C, a, \alpha, \beta > 0$ be some fixed real numbers independent of n. We have*

(1) $n^\alpha = \mathcal{O}(n^\beta)$ *whenever* $\alpha \le \beta$.
(2) $n^C = \mathcal{O}(a^n)$ *for any* $a > 1$.
(3) $(\ln n)^C = \mathcal{O}(n^\alpha)$.

Usnig the smybol $\mathcal{O}()$ we can also write a more exact comparison of two functions.

For example, the notation $f(n) = g(n) + \mathcal{O}(\sqrt{n})$ means that the function f is the same as g up to an "error" of the order $\sqrt{n}$. A simple concrete example is

$$\binom{n}{2} = \frac{1}{2}n^2 + \mathcal{O}(n).$$

We define most common symbols below.

- $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
- $f(n) = \Omega(g(n))$ if $g(n) = \mathcal{O}(f(n))$.
- $f(n) = \Theta(g(n))$ if $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$.
- $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.