Combinatorics and graph theory

Lecture note 4. Combinatorial countings and inclusion exclusion

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We first estimates of the function $n! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$. For small values of n, $n!$ can be very quickly evaluated by a computer. But one might think that the values of the factorial are too large to have any significance in the "real world".

But in various mathematical considerations, we often need to compare the order of magnitude of the function $n!$ to other functions, even for very large values of n.

A very simple thing to try is

$$n! \leq \prod_{1 \leq i \leq n} n \leq n^n,$$

and

$$n! \geq \prod_{1 \leq i \leq n} 2 \geq 2^{n-1}.$$

A simple estimate according to Gauss.
This proof is of some historical interest, since it was invented by the great mathematician Gauss.

Theorem 1. *For every* $n \geq 1$, $n^{n/2} \leq n! \leq (\frac{n+1}{2})^n$.

We use the following lemma

Lemma 1. *For positive real numbers* a *and* b, $\sqrt{ab} \leq \frac{a+b}{2}$.

Proof. It follows from $(\sqrt{a} - \sqrt{b})^2 \geq 0$. □

Proof of Theorem 1.

However, we may keep asking more and more penetrating questions, such as whether $\frac{((n+1)/2)^n}{n!}$ grows to infinity. We will now skip stages, and give more tight bounds.

Theorem 2. *For every $n \geq 1$, we have*

$$e\left(\frac{n}{2}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n.$$

(e is the Euler number 2.718281828...)

We now consider the function $\binom{n}{k}$.

From the definition of $\binom{n}{k}$, we have $\binom{n}{k} \leq n^k$, and for many applications, this simple estimate is sufficient. For $k \geq \frac{n}{2}$, we may use $\binom{n}{k} = \binom{n}{n-k}$.

To derive a lower bound, we look at the definition of the binomial coefficient written as a product of fractions. So then, it is easy to see that

$$\binom{n}{k} \geq \left(\frac{n}{k}\right)^k.$$

Theorem 3. *For every $n \geq 1$ and every $1 \leq k \leq n$, we have*

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

Continue.

Example 1. *The town has* 3 *clubs. The lawn-tennis club has 20 members, the chandelier collector club has 15 members, and the Egyptology club has 8 members. There are 2 tennis players and 3 chandelier collectors among Egyptologists, 6 people both playing tennis and collect chandeliers, and there is even one especially eager person participating in all three clubs. How many people are engaged in the club?*

One might guess that

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = &|A_1| + |A_2| + \cdots |A_n| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_{n-1} \cap A_n| \\ &\cdots \\ &+ (-1)^{n-1}|A_1 \cap A_2 \cap \cdots \cap A_n|.\end{aligned}$$

For a set A and a positive integer k, we denote by $\binom{A}{k}$ the set of all subsets of A of size k.

Theorem 4 (Inclusion-exclusion principle). *For any collection $A_1, A_2, \ldots, A_n$ of finite sets, we have*

$$|\bigcup_{i=1}^{n} A_i| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{I \in \binom{\{1,2,\ldots,n\}}{k}} |\bigcup_{i \in I} A_i|.$$

Proof.

Continue.