Combinatorics and graph theory

Lecture note 5. graphs an introduction

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Many situations in various practically motivated problems and also in mathematics and theoretical computer science can be captured by a scheme consisting of two things:

- a (finite) set of points,
- lines joining some pairs of the points.

Examples:

- Points may represent participants at a birthday party, and the joins correspond to pairs of participants who know each other.
- Points can represent street crossings in a city and the joins the streets.
- Points can represent subway stations and the joins correspond to railways.

Definition 1. *A graph G is an ordered pair (V, E) where V is a set and E is a set of 2-point subsets of V, that is, $E \subseteq V \times V$. The elements of V are called* vertices *and the elements of E are called* edges*.*

Basic notations

- If (u, v) is an edge, we simply denote uv. We also say that u is adjacent to v or u is a neighbor of v.

- Let G and H be two graphs. We say that G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. We say that G is an *induced subgraph* of H if $V(G) \subseteq V(H)$ and $E(G) = E(H) \cap \binom{V(G)}{2}$.

Important graphs

The complete graph K_n.

The cycle C_n.

The path P_n.

The complete bipartite graph $K_{n,m}$.

Bipartite graphs

Graph isomorphism

Definition 2. *Two graphs G and H are* isomorphic *if there exists a bijection $f : V(G) \to V(H)$ such that for all $v, w \in V(G)$ with $v \neq w$*

- $vw \in E(G)$ *if and only if* $vw \in E(H)$.

Such an f is called an isomorphism of G and H, and written by $G \simeq H$.

It can be thought as "renaming the vertices".

Exercise 1. *How many pairwise non-isomorphic graphs on 3 vertices?*

Paths and cycles

A *path* ins a graph is a sequence

$$(v_0, e_1, v_1, \ldots, e_t, v_t)$$

where $v_0, v_1, \ldots, v_t$ are all distinct, and for each $i = \{1, 2, \ldots, t\}$, $e_i = v_{i-1}v_i$ is an edge of G. We also say that it is a path of length t from v_0 to v_t. Note that a path of length 0 consisting of a single vertex.

Similarly, a *cycle* in a graph G is a sequence

$$(v_0, e_1, v_1, \ldots, v_{t-1}, e_t, v_0)$$

(the initial and final points are the same), where $v_0, v_1, \ldots, v_{t-1}$ are all distinct, and for each $i = 1, 2, \ldots, t-1$, $e_i = v_{i-1}v_i$ is an edge and also $e_t = v_{t-1}v_0$ is an edge. The number t is the length of the cycle.

A *walk* ins a graph is a sequence

$$(v_0, e_1, v_1, \ldots, e_t, v_t)$$

where for each $i = \{1, 2, \ldots, t\}$, $e_i = v_{i-1}v_i$ is an edge of G (so it is not necessarily that $v_0, v_1, \ldots, v_t$ are all distinct).

Exercise 2. *Every walk from x to y in a graph contains a path from x to y.*

Connectedness, components

We say that a graph H is *connected* if for any two vertices x and y of H, there is a walk from x to y.

We define a relation $\sim$ on the vertex set $V(G)$ by letting $x \sim y$ if and only if there exists a path from x to y in G. The subgraph of G induced by each equivalent class is called a *connected component* of G.

Distance in graphs

We define the *distance* of two vertices v and w in a graph G, denoted by $d_G(v, w)$, as the length of a shortest path from v to w in G.

The distance function has the following property.

- $d_G(v, v) \geq 0$,
- (symmetry) $d_G(v, w) = d_G(w, v)$,
- (triangle ineqality) $d_G(x, z) \leq d_G(x, y) + d_G(y, z)$ for any three vertices x, y, z.

Adjacency matrix
The *adjacency matrix* A_G of a graph G is the matrix whose rows and columns are indexed by $V(G)$, and for v, w, $A_G[v, w] = 1$ if v is adjacent to w, and $A_G[v, w] = 0$ otherwise.

Proposition 1. *Let A^k be the k-th power of the adjacency matrix $A = A_G$ of G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, and let a^k_{ij} denote the element of the matrix A^k at position (v_i, v_j). Then a^k_{ij} is the number of walks of length exactly k from the vertex v_i to the vertex v_j in the graph G.*

Proof.