Combinatorics and graph theory

Lecture note 6. Eulerian graphs

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Here is one of the oldest problems concerning graph drawing.

Problem. Draw a graph with a single closed line without lifting the pencil from the paper.

Mathematically, this can be formalized as follows: find a closed walk

$$(v_0, e_1, v_1, \ldots, e_{m-1}, v_{m-1}, e_m, v_0)$$

containing all the vertices and all the edges, where it uses each edge exactly once.
Such a walk is called a *closed Eulerian tour*, and a graph possessing a closed Eulerian tour is called *Eulerian.*

Theorem 1. *A graph is Eulerian if and only if it is connected and each vertex has even degree.*

Proof.

Second proof.

Lemma 1. *If a graph G has all degrees even, then the edge set $E(G)$ can be partitioned into disjoint subsets $E_1, E_2, \ldots, E_k$ so that each E_i is the edge set of a cycle.*

Directed graphs

We introduce directed graphs, where every edge has a direction.

Definition 1. *A directed graph G is a pair (V, E), where E is a subset of $V \times V$. The ordered pair $(x, y) \in E$ is called a directed edge. We say that a directed edge (x, y) has head y and tail x.*

It is natural to define a *directed tour* in a directed graph G as a sequence

$$(v_0, e_1, v_1, e_2, \ldots, e_m, v_m)$$

such that $e_i = (v_{i-1}, v_i) \in E$ for each $i = 1, 2, \ldots, m$, and moreover, $e_i \neq e_j$ whenever $i \neq j$. We can define a directed walk, directed path, and directed cycle similarly.

We say that a directed graph is *Eulerian* if it has a closed directed tour containing all vertices and passing each directed edge exactly once.

For a vertex v, the number of edges ending in v is denoted by $\deg_G^+(v)$, and the number of edges originating in v is denoted by $\deg_G^-(v)$.

Proposition 1. *A directed graph G is Eulerian if and only if its symmetrization is connected and $\deg_G^+(v) = \deg_G^-(v)$ for each vertex v of G.*

Applications