Combinatorics and graph theory

Lecture note 7. Triangle-free graphs

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Let us consider a graph G with n vertices. What can be said about the number of its edges?

The answer is easy : $\binom{n}{2}$.

QUESTION : What is the maximum number of edges of a graph with n vertices and no triangles? Let $T(n)$ be the maximum possible number of edges in a graph on n vertices with no triangles.

Clearly, $T(1) = 0$, $T(2) = 1$, and $T(3) = 2$. It is also easy to check that $T(4) = 4$.

What is $T(5)$?

Theorem 1. *For every integer $n \geq 0$, we have* $T(n) = \lfloor \frac{n^2}{4} \rfloor$.

Proof.

We can extract even more from the above proof. We say that a graph $G = (V, E)$ with n vertices is *extremal* for K_3 if it contains no triangle and has $\lfloor \frac{n^2}{4} \rfloor$ edges.

Theorem 2. *For every n every extremal n-vertex graph for K_3 is isomorphic to the graph $K_{a,b}$ with $a = \lfloor \frac{n}{2} \rfloor$ and $b = n - \lfloor \frac{n}{2} \rfloor$.*

Can we generalize this result to K_r for $r > 3$? What is the maximum possible number of edges of a graph with no K_r subgraphs?

An r-partite graph is a graph whose vertex set can be partitioned into r sets $V_1, V_2, \ldots, V_r$ such that all edges lie between two distinct sets of them. It is called a *complete r-partite graph* if all possible edges lie between two parts.

The unique complete r-partite graph on $n \geq r$ vertices whose partition sets differ in size by at most 1 are called *Turán graphs*. We denote them by $T_r(n)$.

We say that a graph $G = (V, E)$ with n vertices is *extremal* for H if it contains no H and has maximum possible number of edges.

Theorem 3. *For all integers r, n with $r > 1$, every extremal n-vertex graph for K_r is $T_{r-1}(n)$.*

PROOF.