Combinatorics and graph theory

Lecture note 8. Drawing graphs

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In order to introduce the notion of a drawing formally we define an *arc*. This is a subset α of the plane of the form $\alpha := \gamma([0,1])$ where $\gamma : [0,1] \to \mathbb{R}^2$ is an injective continuous map of the closed interval. The points $\gamma(0)$ and $\gamma(1)$ are called the *endpoints* of the arc α.

By a drawing of a graph $G = (V, E)$, we mean an assignment as follows:

- to every vertex v, assign a point $b(v)$ of the plane,
- to every edge $\{v, v'\}$, assign an arc $\alpha(e)$ in the plane with endpoints $b(v)$ and $b(v')$.

We assume that the mapping b is injective, and no point of the form $b(v)$ lies on any of the arcs $\alpha(e)$ unless it is an endpoint of that arc.

A drawing of a graph G where any two arcs corresponding to distinct edges either have no intersection or only share an endpoint is called a *planar drawing*. A graph is *planar* if it has a planar drawing. A graph with a fixed planar drawing is called a plane graph (or topological planar graph in this book).

A planar drawing is advantageous for a visualization of a graph, and in some applications where the drawing has a physical meaning.

Faces of a graph drawing
Let $G = (V, E)$ be a plane graph, that is, a planar graph with a fixed drawing.

We say that a set $A \subseteq \mathbb{R}^2$ is *connected* if for any two points $x, y \in A$, there exists an arc $\alpha \subseteq A$ with endpoints x and y. The drawing of G partitioned into connected regions. These regions are called the *faces* of the plane graph.

The region spreading out to infinity is called the *outer face*, and all other regions are called the *inner faces*.

Drawing on other surfaces

Everyone knows the sphere. The surface of a tire-tube is scientifically called the *torus*.

If we take a long strip of paper, turn one of its ends by 180 degrees, and glue it to the other end, we obtain an interesting surface called the *Möbius band*.

Other examples are a sphere with two handles,

or the so-called *Klein bottle*.

Each of these surfaces can be created from a planar polygon by “gluing” and a suitable deformation.

For example, the torus can be made as follows.

More examples:

Graphs can be classified according to the surfces they can be drawn on. As we will be shown in the next section, neither the graph K_5 nor $K_{3,3}$ is planar.

But K_5 can be drawn on the torus, and $K_{3,3}$ can be drawn on the Möbius band.

Proposition 1. *Any graph can be drawn without edge crossing on a sphere with sufficiently many handles.*

The smallest number of handles that must be added to the sphere so that G can be drawn on the resulting surface without edge crossings is called the *genus* of the graph G.

INFORMAL PROOF.