Combinatorics and graph theory

Lecture note 9. Euler's formula

Lecturer : O-joung Kwon
Fall, 2018

There exists essentially a basic formula for planar graphs.

Proposition 1 (Euler's formula)**.** *Let $G = (V, E)$ be a connected planar graph, and let f be the number of faces of some planar drawing of G. Then we have*

$$|V| - |E| + f = 2.$$

PROOF.

Regular polytope
A regular polytope is a 3-dimensional convex body bounded by a finite number of faces, such that all faces are congruent copies of the same regular convex polygon, and the same number of faces meet at each vertex of the body.

There are only 5 types of regular polytopes. Why?

Using Euler's formula, we can show this. Remark that if we relax the conditions on regularity a little, or go to higher dimensions, we have more examples.

We can project a polytope on the sphere without edge crossings, and project to a plane in a natural way, so called stereographic projection.

Proposition 2. *Let G be a plane graph where each vertex has degree d and each face is adjacent to k vertices, for some integers $d \geq 3$ and $k \geq 3$. Then G is isomorphic to one of 5 graphs above mentioned.*

PROOF.

A very important property of planar graphs is that they can only have relatively few edges: a planar graph on n vertices has $\mathcal{O}(n)$ edges.

Proposition 3. *Let $G = (V, E)$ be a planar graph with at least 3 vertices. Then*

(1) $|E| \leq 3|V| - 6$.
(2) Equaliaty hold for any maximal planar graphs; that is a palanr graph such that adding any new edge (while preserving the same vertex set) makes it non-planar.

PROOF.

We show that K_5 and $K_{3,3}$ are non-planar.

Proposition 4. *K_5 and $K_{3,3}$ are non-planar.*