Abstract algebra

Lecture note 10. Factor group computation and simple groups

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Let N be a normal subgroup of G. We may consider G/N as a group obtained by collapsing each coset to an element.

Example 1. The trivial subgroup $N = \{0\}$ of $\mathbb{Z}$ is a normal subgroup. Compute $\mathbb{Z}/\{0\}$.

Example 2. Let n be a positive integer. The set $n\mathbb{R} = \{nr : r \in \mathbb{R}\}$ is a subgroup of $\mathbb{R}$ under addition, and it is normal. Compute $\mathbb{R}/n\mathbb{R}$.

As illustrated these examples, we can see that $G/\{e\} \simeq G$ and $G/G \simeq \{e\}$.

We give two examples showing that G/N has order 2, but has interesting properties.

Example 3. Because $|S_n| = 2|A_n|$, A_n is a normal subgroup of S_n, and S_n/A_n has order 2. This factor group is isomorphic to $\mathbb{Z}_2$.

Example 4. (Falsity of the Converse of the Theorem of Lagrange)

We look at that if the original group was finitely generated and abelian, then its factor group will be also.

Example 5. Compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/<(0,1)>$.

Theorem 1. *Let $G = H \times K$ be the direct product of groups H and K. Then $\bar{H} = \{(h,e) : h \in H\}$ is a normal subgroup of G. Also $G/\bar{H}$ is isomorphic to K.*

Theorem 2. *A factor group of a cyclic group is cyclic.*

PROOF

Example 6. Compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/ < (0, 2) >$.

Definition 1. *A group is* simple *if it is nontrivial and has no proper nontrivial normal subgroups.*

It is known that the alternating subgroup A_n is simple for $n \geq 5$. (This is actually very important in the Galois Theory, but we will not discuss at the moment.)

We characterize those normal subgroups N such that G/N is a simple group.

Theorem 3. *Let $\phi : G \to G'$ be a group homomorphism. If N is a normal subgroup of G, then $\phi[N]$ is a normal subgroup of $\phi[G]$. Also, if N' is a normal subgroup of $\phi[G]$, then $\phi^{-1}[N']$ is a normal subgroup of G.*

Definition 2. *A maximal normal subgroup of a group G is a normal subgroup M not equal to G such that there is no proper normal subgroup N of G properly containing M.*

Theorem 4. *M is a maximal normal subgroup of G if and only if G/M is simple.*

PROOF