Abstract algebra

Lecture note 11. Group action on a set

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For a group G and a set X, we consider a map from $* : G \times X \to X$. We shall write $*(g, x)$ as $g * x$ or gx.

Definition 1. *Let X be a set and G a group. An action of G on X is a map $* : G \times X \to X$ such that*

- *$ex = x$ for all $x \in X$,*
- *$(g_1 g_2)(x) = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$.*

We will call X a G-set.

Example 1. Let X be any set and H be a subgroup of the group S_X of all permutations of X. Then X is an H-set, where the action of $\sigma \in H$ on X is its action as an element of S_X, so that $\sigma x = \sigma(x)$ for all $x \in X$.

Theorem 1. *Let X be a G-set. For each $g \in G$, the function $\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$ for $x \in X$ is a permutation of X. Also, the map $\phi : G \to S_X$ defined by $\phi(g) = \sigma_g$ is a homomorphism with the property that $\phi(g)(x) = gx$.*

PROOF.

Example 2. Considering group as a set, every group G is itself a G-set, where the action on $g_2 \in G$ by $g_1 \in G$ is $g_1 g_2$.

Example 3. Let H be a subgroup of G. Then G is an H-set under conjugation where $*(h, g) = hgh^{-1}$ for $g \in G$ and $h \in H$.

Example 4. The set of vectors is an $\mathbb{R}^*$-set for the multiplicative group of non-zero scalars.

Example 5. X be the points in a rectangle (16.9). X can be regarded as a D_4-set.

Isotropy subgroups
Let X be a G-set. Let $x \in X$ and $g \in G$. It will be important to know when $gx = x$. We let

- $X_g = \{x \in X : gx = x\}$ and $G_x = \{g \in G : gx = x\}$.

Theorem 2. *Let X be a G-set. Then G_x is a subgroup of G for each $x \in X$.*

Definition 2. *This subgroup is called the isotropy subgroup of x.*

PROOF.

Orbits

Theorem 3. *Let X be a G-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Then $\sim$ is an equivalence relation on X.*

Definition 3. *Let X be a G-set. Each cell in the partition of the equivalence relation described in the above theorem is an orbit in X under G. If $x \in X$¡ the cell containing x is the orbit of x, call it Gx.*

Theorem 4. *Let X be a G-set and $x \in X$. Then $|Gx| = (G : G_x)$. If $|G|$ is finite, then $|Gx|$ is a divisor of $|G|$.*

PROOF