Abstract algebra

Lecture note 12. Applications of G-sets to counting

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Spring, 2019

The following theorem gives a tool for determining the number of orbits in a G-set X under G. Recall that for each $g \in G$, we let X_g be the set of elements of X left fixed by g, so that $X_g = \{x \in X : gx = x\}$. Recall that for each $x \in X$, we let $G_x = \{g \in G : gx = x\}$, and Gx is the orbit of x under G.

Theorem 1 (Burnside's Formula)**.** *Let G be a finite group and X a finite G-set. If r is the number of orbits in X under G, then we have*

$$r|G| = \sum_{g \in G} |X_g|.$$

PROOF.

Corollary 1. *Let G be a finite group and X a finite G-set. The number of orbits in X under G equals*

$$\frac{1}{|G|}\sum_{g\in G}|X_g|.$$

Example 1. Let X be the set of 720 different markings of a cube using from one to six dots. Let G be the group of 24 rotations. So, the number of orbits is 30.

Example 2. Considering group as a set, every group G is itself a G-set, where the action on $g_2 \in G$ by $g_1 \in G$ is $g_1 g_2$.

Example 3. Let H be a subgroup of G. Then G is an H-set under conjugation where $*(h, g) = hgh^{-1}$ for $g \in G$ and $h \in H$.

Example 4. The set of vectors is an $\mathbb{R}^*$-set for the multiplicative group of non-zero scalars.

Example 5. X be the points in a rectangle (16.9). X can be regarded as a D_4-set.

Isotropy subgroups

Let X be a G-set. Let $x \in X$ and $g \in G$. It will be important to know when $gx = x$. We let

- $X_g = \{x \in X : gx = x\}$ and $G_x = \{g \in G : gx = x\}$.

Theorem 2. *Let X be a G-set. Then G_x is a subgroup of G for each $x \in X$.*

Definition 1. *This subgroup is called the isotropy subgroup of x.*

PROOF.

Orbits

Theorem 3. *Let X be a G-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Then $\sim$ is an equivalence relation on X.*

Definition 2. *Let X be a G-set. Each cell in the partition of the equivalence relation described in the above theorem is an orbit in X under G. If $x \in X$¡ the cell containing x is the orbit of x, call it Gx.*

Theorem 4. *Let X be a G-set and $x \in X$. Then $|Gx| = (G : G_x)$. If $|G|$ is finite, then $|Gx|$ is a divisor of $|G|$.*

PROOF