Abstract algebra

Lecture note 13. Sylow theorems

Lecturer : O-joung Kwon

Spring, 2019

The study of finite nonabelian groups is much more complicated. The Sylow theorems give us some important information about them.

If G is abelian, then there exist subgroups of every order dividing $|G|$. We show that this is not true for non-abelian group.

p-groups
Let X be a finite G-set, and recall $Gx = \{gx : g \in G\}$. Suppose that there are r-orbits in X under G, say $\{x_1, x_2, \ldots, x_r\}$ contain one element from each orbit. We know that $|X| = \sum_{i=1}^{r} |Gx_i|$.

There may be one-element orbits. Let $X_G = \{x \in X : gx = x \text{ for all } g \in G\}$. Thus, X_G is precisely the union of the ono-element orbits, and we may rewrite as

$$|X| = |X_G| + \sum_{i=1}^{r} |Gx_i|.$$

Theorem 1. *Let G be a group of order p^n and X be a finite G-set. Then $|X| \equiv |X_G| \pmod{p}$.*

Definition 1. *Let p be a prime. A group G is a p-group if every element in G has order a power of the prime p. A subgroup of a group G is a p-subgroup if the subgroup is a p-group.*

Theorem 2 (Cauchy's theorem)**.** *Let p be a prime. Let G be a finite group and let p divide $|G|$. Then G has an element of order p, and consequently, a subgroup of order p.*

Corollary 1. *Let G be a finite group. Then G is a p-group if and only if $|G|$ is a power of p.*

PROOF.

Sylow Theorems

Let G be a group, and $\mathcal{T}$ be the collection of all subgroups of G. We make $\mathcal{T}$ into a G-set by letting G act on $\mathcal{T}$ by conjugation gHg^{-1}.

Now, $G_H = \{g \in G : gHg^{-1} = H\}$ is easily seen to be a subgroup of G, and H is a normal subgroup of G_H. Since G_H consists of all elements of G that leave H invariant under conjugation, G_H is the largest subroup of G having H as a normal subgroup.

Definition 2. *The subgroup G_H is the normalizer of H in G, and denoted by $N[H]$.*

Lemma 1. *Let H be a p-subgroup of a finite group G. Then*

$$(N[H] : H) \equiv (G : H) \pmod{p}.$$

Corollary 2. *If p divides $(G : H)$, then $N[H] \neq H$.*

PROOF.

Theorem 3 (First Sylow Theorem)**.** *Let G be a finite group with $|G| = p^n m$ where $n \geq 1$ and p does not divide m. Then*

(1) G contains a subgroup of order p^i for each $1 \leq i \leq n$.
(2) Every subgroup H of G of order p^i is a normal subgroup of a subgroup of order p^{i+1} for $1 \leq i < n$.

PROOF.

Definition 3. *A Sylow p-subgroup P of a group G is a maximal p-subgroup of G, that is, a p-subgroup contained in no lager p-subgroup.*

If $|G| = p^n m$, then the Sylow p-subgroup of G are subgroups of order p^n. The next theorem shows that two Sylow p-subgroups are conjugate.

Theorem 4 (Second Sylow Theorem)**.** *Let P_1 and P_2 be Sylow p-subgroups of a finite group G. Then P_1 and P_2 are conjugate subgroups of G.*

Theorem 5 (Third Sylow Theorem)**.** *If G is a finite group and p divides $|G|$, then the number of Sylow p-subgroups is congruent to* 1 *modulo p and divides $|G|$.*

Example.