Abstract algebra

Lecture note 4. Groups of permutations

Lecturer : O-joung Kwon
Spring, 2019

We have seen examples of groups like $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ under addition. Also groups of matrices like $GL(2, \mathbb{R})$. Each element of $GL(2, \mathbb{R})$ can be seen as operation that moves 2-dimensional vector to a 2-dimensional vector.

In this section, we consider permutations that act on a finite set.

Definition 1. *A permutation of a set A is a function $\phi : A \to A$ that is both one to one and onto.*

We consider a function composition $\circ$ that is a binary operation on the collection of all permutations of a set A. Let σ and τ be permutations of A. The *composition* $\sigma \circ \tau$ defined schematically by

$$A \to (\tau)A \to (\sigma)A.$$

Rather than keeping $\circ$, we stick two permutations, like $\sigma\tau$ (we apply right to left order)

Clearly, $\sigma\tau$ is a permutation.

Example 1. Suppose $A = \{1, 2, 3, 4, 5\}$. A permutation σ can be defined as

$$\sigma(1) = 4, \sigma(2) = 2, \sigma(3) = 5, \sigma(4) = 3, \sigma(5) = 1.$$

We arite σ in a more standard notation, changing the columns to rows in parentheses and omitting the

arrows, as $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix}$

Let $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{pmatrix}$

Then $\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 2 & 4 \end{pmatrix}$

Theorem 1. *Let A be a set and S_A be the collection of all permutations of A. Then S_A is a group under permutation multiplication.*

PROOF

Definition 2. *Let $A = \{1, 2, \ldots, n\}$. The group of all permutations of A is the symmetric group on n letters, and is denoted by S_n. Note that S_n has $n!$ elements.*

Two important examples

Example 2. An interesting example is S_3. It can be seen as reflecting or rotating a triangle with labels $1, 2, 3$. It is not abelian (nonabelian).

Example 3. The n-th dihedral group D_n is the group of symmetries of the regular n-gon. The order is $2n$. (reflection + rotation n)

Observe that S_3 and D_3 are isomorphic.

If you see a table of a group, a column gives a permutation of group elements. In view of these, it is not surprising that every finite group G is isomorphic to a subgroup of the group S_G of all permutations of G.

Definition 3. *Let $f : A \to B$ be a function and H be a subset of A. The image of H under f is $\{f(h) : h \in H\}$ and is denoted by $f(H)$.*

Lemma 1. *Let G and G' be groups and $\phi : G \to G'$ be a one-to-one function such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. Then $\phi(G)$ is a subgroup of G' and ϕ provides an isomorphism of G with $\phi[G]$.*

PROOF

Theorem 2 (Cayley's theorem)**.** *Every group is isomorphic to a group of permutations.*

PROOF