Abstract algebra

Lecture note 6. Cosets and Lagrange theorem

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In this section, we mainly show that for every subgroup H of a group G, $|H|$ divides $|G|$. We shall prove it by providing a partition of G into cells, all having the same size as H.

Cosets

Theorem 1. *Let H be a subgroup of G. Let the relation $\sim_L$ be defined on G by*

- $a \sim_L b$ *if and only if* $a^{-1}b \in H$.

Similarly, let the relation $\sim_R$ be defined on G by

- $a \sim_R b$ *if and only if* $ab^{-1} \in H$.

Then $\sim_L$ and $\sim_R$ are both equivalence relations on G.

PROOF.

Definition 1. *Let H be a subgroup of a group G. Then the subset $aH = \{ah : h \in H\}$ of G is the left coset of H containing a, while the subset $Ha = \{ha : h \in H\}$ is the right coset of H containing a.*

Example 1. Exhibit left cosets and right cosets of the subgroup $3\mathbb{Z}$ of $\mathbb{Z}$.

- For a subgroup H of an abelian group G, the partition of G into left cosets of H and the partition into right cosets are the same.

Lemma 1. *Every coset aH of a subgroup H of a group G has the same size as H.*

PROOF.

We now prove the theorem of Lagrange.

Theorem 2 (Theorem of Lagrange)**.** *Let H be a subgroup of a finite group G. Then the order of H is a divisor of the order of G.*

Corollary 1. *Every group of prime order is cyclic.*

Theorem 3. *The order of an element of a finite group divides the order of the group.*

Definition 2. *Let H be a subgroup of a group G. The number of left cosets of H in G is the* index *$(G : H)$ of H in G.*

If G is finite, then obviously, $(G : H)$ is finite and $(G : H) = \frac{|G|}{|H|}$. (Exercise) The index could be equally well defined as the number of right cosets of H in G.

Theorem 4. *Suppose H and K are subgroups of G such that $K \leq H \leq G$, and suppose $(H : K)$ and $(G : H)$ are both finite. Then $(G : K)$ is finite and $(G : K) = (G : H)(H : K)$.*