Abstract algebra

Lecture note 7. Direct products

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Our purpose of this section is to show a way to use known groups as building blocks to form more groups.

Definition 1. *The* Cartesian product *of sets* $S_1, S_2, \dots, S_n$ *is the set of all ordered* n*-tuples* $(a_1, a_2, \dots, a_n)$ *where* $a_i \in S_i$ *for* $i = 1, 2, \dots, n$. *The Cartesian product is denoted by either* $S_1 \times S_2 \times \cdots \times S_n$ *or by* $\prod_{i=1}^{n} S_i$.

Theorem 1. *Let* $G_1, G_2, \dots, G_n$ *be groups. For* $(a_1, a_2, \dots, a_n)$ *and* $(b_1, b_2, \dots, b_n)$ *in* $\prod_{i=1}^{n} G_i$, *define* $(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n) = (a_1 b_1, a_2 b_2, \dots, a_n b_n)$. *Then* $\prod_{i=1}^{n} G_i$ *is a group, called the* direct product of the groups G_i, *under this binary operation.*

Sometimes, we use additive notation in $\prod_{i=1}^n G_i$ and refer to as the *direct sum of the groups* G_i. The notation $\oplus_{i=1}^n G_i$ is somtimes used.

Example 1. Consider $\mathbb{Z}_2 \times \mathbb{Z}_3$, which has 6 elements. We claim that it is cyclic ($(1,1)$ is a generator).

Example 2. Consider $\mathbb{Z}_3 \times \mathbb{Z}_3$. It is not cyclic.

Theorem 2. *The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime, that is, $gcd(m, n) = 1$.*

PROOF.

Corollary 1. *The group* $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ *is cyclic and isomorphic to* $\mathbb{Z}_{m_1 m_2 \cdots m_n}$ *if and only if gcd of any two of* $m_1, m_2, \ldots, m_n$ *is 1.*

Theorem 3. *Let* $(a_1, a_2, \ldots, a_n) \in \prod_{i=1}^{n} G_i$. *If* a_i *is of finite order* r_i *in* G_i, *then the order of* $(a_1, a_2, \ldots, a_n)$ *in* $\prod_{i=1}^{n} G_i$ *is equal to* $lcm(r_1, r_2, \ldots, r_n)$.

Example 3. Find the order of $(8, 4, 10)$ in the group $\mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$.

Theorem 4 (Fundamental theorem of finitely generated abelian groups)**.** *Every finitely generated abelian group G is isomorphic to a directe product of cyclic groups in the form*

$$\mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$$

where p_i's are primes and r_i's are positive integers. The direct product is unique except for possible rearrangement of the factors.

Example 4. Find all abelian groups, up to isomorphism, of order 360.

Theorem 5. *The finite indecomposable abelian groups are exactly the cyclic groups with order a power of a prime.*