Abstract algebra

Lecture note 8. Homomorphisms

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Spring, 2019

We are interested in maps from G to G" that relate the group structure of G to the group structure of G'.

Definition 1. *A map ϕ of a group G into a group G' is a* homomorphism *if the homomorphism property $\phi(ab) = \phi(a)\phi(b)$ holds for all $a, b \in G$.*

A homomorphism is trivial if $\phi(g) = e'$ for all $g \in G$ where e' is the identity of G'.

Example 1. Let $\phi : G \to G'$ be a group homomorphism of G onto G'. If G is abelian, then G' is abelian.

Example 2. Let S_n be the symmetric group on n letters, and let $\phi : S_n \to \mathbb{Z}_2$ be defined by

- $\phi(\sigma) = 0$ if σ is an even permutation,
- $\phi(\sigma) = 1$ if σ is an odd permutation,

Then ϕ is a homomorphism.

Example 3. (Evaluation homomorphism) Let F be the additive group of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, and let $\mathbb{R}$ be the additive group of real numbers. Let c be any real number. Let $\phi_c : F \to \mathbb{R}$ be the evaluation homomorphism defined by $\phi_c(f) = f(c)$.

Example 4. Let A be an $m \times n$ matrix of real numbers. Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be defined by $\phi(v) = Av$. Then ϕ is a homomorphism.

Example 5. Let $GL(n, \mathbb{R})$ be the multiplicative group of all invertible $n \times n$ matrices. Note that $\det(AB) = \det(A)\det(B)$. So, det is a homomorphism mapping $GL(n, \mathbb{R})$ into the multiplicative group $\mathbb{R}^*$ of nonzero real numbers.

Properties of homomorphisms

Definition 2. *Let ϕ be a mapping of a set X into a set Y, and $A \subseteq X$ and $B \subseteq Y$. The image $\phi[A]$ of A in Y under ϕ is $\{\phi(a) : a \in A\}$. The set $\phi[X]$ is the range of ϕ. The inverse image $\phi^{-1}[B]$ of B in X is $\{x \in X : \phi(x) \in B\}$.*

Theorem 1. *Let ϕ be a homomorphism of a group G into a group G'. The following are true.*

(1) *If e is the identity element in G, then $\phi(e)$ is the identity element e' in G'.*
(2) *If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$.*
(3) *If H is a subgroup of G, then $\phi[H]$ is a subgroup of G'.*
(4) *If K' is a subgroup of G', then $\phi^{-1}[K']$ is a subgroup of G.*

Definition 3. *Let $\phi : G \to G'$ be a homomorphism of groups. The subgroup $\phi^{-1}[\{e'\}]$ is the kernel of ϕ, denoted by $Ker(\phi)$.*

Theorem 2. *Let $\phi : G \to G'$ be a homomorphism of groups, and $H = Ker(\phi)$. Let $a \in G$. Then the set $\phi^{-1}[\phi(a)] = \{x \in G : \phi(x) = \phi(a)\}$ is the left coset aH of H, and is also the right coset Ha of H. Consequently, the two partitions of G into left cosets and into right cosets are the same.*

Corollary 1. *A group homomorphism* $\phi : G \to G'$ *is a one-to-one map if and only if* $Ker(\phi) = \{e\}$.