Abstract algebra

Lecture note 9. Factor groups

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Theorem 1. *Let $\phi : G \to G'$ be a homomorphism of groups with kernel H. Then the cosets of H form a* factor group*, denoted by G/H, where $(aH)(bH) = (ab)H$. Also the map $\mu : G/H \to \phi[G]$ defined by $\mu(aH) = \phi(a)$ is an isomorphism.*

A factor group G/H is often called the factor group of G modulo H.

Example 1. We consider a map $\gamma : \mathbb{Z} \to \mathbb{Z}_m$, where $\gamma(m)$ is the remainder when m is divided by n. Of course $Ker(\gamma) = n\mathbb{Z}$. By Theorem 1, we see that the factor group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$.

$(2 + 5\mathbb{Z}) + (4 + 5\mathbb{Z}) = 6 + 5\mathbb{Z} = 1 + 5\mathbb{Z}$.

General factor group So far, we defined factor groups from homomorphisms. To define it without homomorphism, we need the notion of normal subgroups.

Definition 1. *A subgroup H of a group G is* normal *if its left coset and right cosets coincide, that is, $gH = Hg$ for all $g \in G$.*

Theorem 2. *Let H be a subgroup of a group G. Then left coset multiplication is well defined by $(aH)(bH) = (ab)H$ if and only if H is a normal subgroup of G.*

Corollary 1. *Let H be a normal subgroup of G. Then the cosets of H form a group G/H under the binary operation $(aH)(bH) = (ab)H$.*

This factor group is the factor group (or quotient group) of G by H.

Example 1. For an abelian group G, any subgroup is a normal subgroup.

The fundamental homomorphism theorem We have seen that every homomorphism ϕ gives rise to a natural factor group $G/Ker(\phi)$. We can show that each factor group G/H gives rise to a natural homomorphism having H as a kernel.

Theorem 3. *Let H be a normal subgroup of G. Then $\gamma : G \to G/H$ given by $\gamma(x) = xH$ is a homomorphism with kernel H.*

Theorem 4 (Fundamental homomorphism theorem)**.** *Let $\phi : G \to G'$ be a group homomorphism with kernel H. Then $\phi[G]$ is a group, and $\mu : G/H \to \phi[G]$ given by $\mu(gH) = \phi(g)$ is an isomorphism. If $\gamma : G \to G/H$ is the homomorphism given by $\gamma(g) = gH$, then $\phi(g) = \mu\gamma(g)$ for each $g \in G$.*

Example 2. Classify the group $(\mathbb{Z}_4 \times \mathbb{Z}_2)/(\{0\} \times \mathbb{Z}_2)$.

Alternative characterization of normal subgroups

Theorem 5. *The following are three equivalent conditions for a subgroup* H *of a group* G *to be a normal subgroup.*

(1) $ghg^{-1} \in H$ *for all* $g \in G$ *and* $h \in H$.
(2) $gHg^{-1} = H$ *for all* $g \in G$.
(3) $gH = Hg$ *for all* $g \in G$.

Definition 2. *An isomorphism* $\phi : G \to G$ *of a group* G *with itself is an automorphism of* G. *The automorphism* i_g *where* $i_g(x) = gxg^{-1}$ *for all* $x \in G$, *is the inner automorphism of* G *by* g. *Performing* i_g *on* x *is called conjugation of* x *by* g.

Normal subgroups of a group are precisely those that are invariant under all inner automorphisms. A subgroup K *of* G *is a conjugate subgroup of* H *if* $K = i_g[H]$ *for some* $g \in G$.